Stability and symmetry breaking in the general n-Higgs-doublet model

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For potentials with n-Higgs-boson doublets stability, electroweak symmetry breaking, and the stationarity equations are discussed in detail. This is done within the bilinear formalism which simplifies the investigation, in particular since irrelevant gauge degrees of freedom are systematically avoided. For the case that the potential leads to the physically relevant electroweak symmetry breaking the mass matrices of the physical Higgs bosons are given explicitly.

1. INTRODUCTION

Despite the fact that the Standard Model (SM) has only one Higgs-boson doublet, there is no theoretical restiction to impose a larger number of Higgs-boson doublets. In particular, an extended Higgs sector opens the possibility of CP violation in the Higgs potential. This was already shown by T.D. Lee for the case of the two-Higgs-doublet model (THDM) [1].

Here we want to focus on the general n-Higgs-doublet potential, where we assume that all doublets carry the same hypercharge. The aim is to find precise conditions for stability, electroweak symmetry-breaking, as well as to give equations to find systematically all stationary points, in particular, to detect the global minimum. It was shown that this is indeed possible in the case of the THDM [2] as well as in the 3HDM [3]. Here we want to generalise these findings. We will apply the *bilinear* formalism, which was developed in [2, 4] and independently in [5]. Let us note, that the one-to-one correspondance of the gauge orbits of the Higgs-boson doublets and the bilinears in the general nHDM was already given in [2].

On the experimental side there is lots of effort spent to detect more than one physical Higgs boson for instance by the current LHC experiments. On the theoretical side also many models have been proposed which involve an extended Higgs sector. It is well known that supersymmetric models like the minimal and the next-to-minimal supersymmetric standard model require extended Higgs sectors. For reviews see for instance [6] and [7, 8], respectively. Two-Higgs-doublet models have been reviewed in [9]. The completely general Higgs sector was considered in [10] in connection with possible CP-violating effects in Z-boson decays. For further works on models with extended Higgs sectors see for instance [11–17]. Interesting relations between charge breaking, CP violating, and the normal vacuum in multi-Higgs-doublet models were obtained in [18]. Let us also mention that various aspects of the general nHDM in terms of bilinears have been discussed in [2, 5, 19–24].

2. BILINEARS

Let us now consider the tree-level Higgs potential of models with n Higgs-boson doublets satisfying $SU(2)_L \times U(1)_Y$ electroweak gauge symmetry. The case of n Higgs-boson doublets is a generalisation of the cases with two or three doublets which were discussed in detail in [2, 3].

We will assume that we have $n \geq 2$ doublets which all carry the same hypercharge $y = +1/2$ and denote the complex doublet fields by

$$\varphi_i(x) = \begin{pmatrix} \varphi_i^+(x) \\ \varphi_i^0(x) \end{pmatrix}; \qquad i = 1, \ldots, n. \tag{2.1}$$

The most general $SU(2)_L \times U(1)_Y$ gauge invariant Higgs potential consists solely of products of the Higgs-boson doublets in the form

$$\varphi_i(x)^\dagger \varphi_j(x), \qquad i, j \in \{1, \ldots, n\}. \tag{2.2}$$

We will now introduce gauge invariant bilinears, which turn out to be convenient to discuss the properties of the Higgs potential such as its stability and its stationary points.

To this end we introduce the $n \times 2$ matrix of the Higgs-boson fields

$$\phi = \begin{pmatrix} \varphi_1^+ & \varphi_1^0 \\ \vdots & \vdots \\ \varphi_n^+ & \varphi_n^0 \end{pmatrix} = \begin{pmatrix} \varphi_1^{\mathrm{T}} \\ \vdots \\ \varphi_n^{\mathrm{T}} \end{pmatrix}. \tag{2.3}$$

All possible $SU(2)_L \times U(1)_Y$ invariant scalar products may be arranged into the hermitian $n \times n$ matrix

$$\underline{K} = \phi\phi^\dagger = \begin{pmatrix} \varphi_1^\dagger\varphi_1 & \varphi_2^\dagger\varphi_1 & \cdots & \varphi_n^\dagger\varphi_1 \\ \varphi_1^\dagger\varphi_2 & \varphi_2^\dagger\varphi_2 & \cdots & \varphi_n^\dagger\varphi_2 \\ \vdots & & \ddots & \vdots \\ \varphi_1^\dagger\varphi_n & \varphi_2^\dagger\varphi_n & \cdots & \varphi_n^\dagger\varphi_n \end{pmatrix}. \tag{2.4}$$

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A basis for the $n \times n$ matrices is given by the n^2 matrices

$$\lambda_\alpha, \quad \alpha = 0, 1, \ldots, n^2 - 1 \tag{2.5}$$

where

$$\lambda_0 = \sqrt{\frac{2}{n}} \mathbb{1}_n \tag{2.6}$$

is the conveniently scaled unit matrix and λ_a, $a = 1, \ldots, n^2 - 1$, are the generalised Gell-Mann matrices. An explicit construction and numbering scheme of the generalised Gell-Mann matrices is given in appendix A. We will here and in the following assume that greek indices (α, β, ...) run from 0 to $n^2 - 1$ and latin indices (a, b, ...) from 1 to $n^2 - 1$. We find

$$\begin{aligned} \mathrm{tr}(\lambda_\alpha \lambda_\beta) &= 2\delta_{\alpha\beta}, \\ \mathrm{tr}(\lambda_\alpha) &= \sqrt{2n}\, \delta_{\alpha 0}. \end{aligned} \tag{2.7}$$

The decomposition of $\underline{K}$ (2.4) reads now

$$\underline{K} = \frac{1}{2} K_\alpha \lambda_\alpha \tag{2.8}$$

where the real coefficients K_α are given by

$$K_\alpha = K_\alpha^* = \mathrm{tr}(\underline{K} \lambda_\alpha). \tag{2.9}$$

Note that in particular

$$K_0 = \mathrm{tr}(\underline{K} \lambda_0) = \sqrt{\frac{2}{n}} \left(\varphi_1^\dagger \varphi_1 + \ldots + \varphi_n^\dagger \varphi_n \right). \tag{2.10}$$

With the matrix $\underline{K}$, as defined in terms of the doublets in (2.4), as well as the decomposition (2.8), (2.9), we may immediately express the scalar products in terms of the bilinears.

The matrix $\underline{K}$ (2.4) is positive semidefinite which follows directly from its definition $\underline{K} = \phi \phi^\dagger$. The n^2 coefficients K_α of its decomposition (2.8) are completely fixed given the Higgs-boson fields.

The $n \times 2$ matrix ϕ has trivially rank smaller or equal 2, from which follows that this holds also for the matrix $\underline{K}$. As was shown in detail in [2] (see their theorem 5), any hermitian $n \times n$ matrix with rank equal or smaller than 2 determines the Higgs-boson fields φ_i, $i = 1, \ldots, n$ uniquely, up to a gauge transformation.

Let us now discuss the properties of the matrix $\underline{K}$ with respect to its rank. Since the $n \times n$ matrix $\underline{K}$ is hermitian and positive semidefinite we can, by a unitary transformation U, diagonalise this matrix,

$$U \underline{K} U^\dagger = \begin{pmatrix} \kappa_1 & 0 & \cdots & 0 \\ 0 & \kappa_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \kappa_n \end{pmatrix}, \tag{2.11}$$

with all $\kappa_i \geq 0$. We define for any hermitian matrix $\underline{K}$ with eigenvalues $\kappa_1, \ldots, \kappa_n$ the symmetric sums

$$\begin{aligned} s_0 &:= 1, \\ s_1 &:= \kappa_1 + \kappa_2 + \cdots + \kappa_n, \\ s_2 &:= \sum_{1 \leq i_1 < i_2 \leq n} \kappa_{i_1} \kappa_{i_2}, \\ s_k &:= \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \kappa_{i_1} \kappa_{i_2} \cdots \kappa_{i_k}, \\ s_n &:= \kappa_1 \cdot \kappa_2 \cdot \ldots \cdot \kappa_n = \det(\underline{K}). \end{aligned} \tag{2.12}$$

The hermitian matrix $\underline{K}$ is positive semidefinite if and only if

$$s_k \geq 0 \qquad \text{for } k = 0, \ldots, n. \tag{2.13}$$

Suppose the matrix $\underline{K}$ has rank 0, then, clearly, all κ_i have to vanish, corresponding to

$$s_1 = s_2 = \ldots s_n = 0. \tag{2.14}$$

Vice versa, starting with the conditions (2.14) for a hermitian matrix $\underline{K}$, the last condition $s_n = 0$ requires that one eigenvalue has to vanish, for instance, $\kappa_n = 0$, without loss of generality. The next-to-last condition in turn requires that another, say $\kappa_{n-1} = 0$, and so on. Therefore we get $\underline{K} = 0$.

Next suppose the hermitian matrix $\underline{K}$ has rank 1, then, without loss of generality, we can assume

$$\begin{aligned} &\kappa_1 > 0, \\ &\kappa_2 = \ldots = \kappa_n = 0. \end{aligned} \tag{2.15}$$

If follows immediately from (2.12)

$$\begin{aligned} &s_1 > 0, \\ &s_2 = \ldots = s_n = 0. \end{aligned} \tag{2.16}$$

On the other hand, having the conditions (2.16) for a hermitian matrix $\underline{K}$ fulfilled, employing (2.12), the last condition $s_n = 0$ requires that at least one κ_i vanishes, for instance $\kappa_n = 0$ without loss of generality. Then the next-to-last condition requires that another eigenvalue has to vanish, for instance $\kappa_{n-1} = 0$. That is, we have $\kappa_n = \ldots = \kappa_2 = 0$. Eventually, the first condition dictates that $\kappa_1 > 0$, hence, $\underline{K}$ has rank 1 and is positive semidefinite.

Suppose the hermitian matrix $\underline{K}$ has rank 2, then, without loss of generality, we can assume

$$\begin{aligned} &\kappa_1 > 0, \kappa_2 > 0, \\ &\kappa_3 = \ldots = \kappa_n = 0. \end{aligned} \tag{2.17}$$

If follows immediately from (2.12)

$$\begin{aligned} &s_1 > 0, s_2 > 0, \\ &s_3 = \ldots = s_n = 0. \end{aligned} \tag{2.18}$$

On the other hand, having the conditions (2.18) for a hermitian matrix $\underline{K}$ fulfilled, employing (2.12), the conditions $s_3 = \ldots = s_n = 0$ require that $\kappa_3 = \ldots = \kappa_n = 0$, without loss of generality. Then the first two conditions of (2.18) state $\kappa_1 + \kappa_2 > 0$ and $\kappa_1 \cdot \kappa_2 > 0$, that is, we have $\kappa_1 > 0$ and $\kappa_2 > 0$. Hence, $\underline{K}$ has rank 2 and is positive semidefinite.

Therefore, we have shown the following theorem.

Theorem 1. *Let $\underline{K} = K_\alpha \lambda_\alpha / 2$ be a hermitian matrix. $\underline{K}$ has rank k with $k = 0, 1, 2$ and is positive semidefinite if and only if*

$$\begin{aligned} &s_0 > 0, \ldots, s_k > 0, \\ &s_{k+1} = \ldots = s_n = 0. \end{aligned} \tag{2.19}$$

We may express the symmetric sums s_k defined in (2.12) in terms of basis-independent traces of powers of $\underline{K}$. We have a recursion relation:

$$\begin{aligned} s_0 &= 1 \\ s_k &= \frac{1}{k} \sum_{i=1}^{k} (-1)^{i-1} s_{k-i} \operatorname{tr}(\underline{K}^i), \qquad k = 1, \ldots, n \,. \end{aligned} \tag{2.20}$$

The derivation of (2.20) is given in appendix A. Explicitly we get for $k = 1, 2, 3$,

$$\begin{aligned} s_1 =& \operatorname{tr}(\underline{K}) = \sqrt{\frac{n}{2}} K_0, \\ s_2 =& \frac{1}{2} \left(\operatorname{tr}^2(\underline{K}) - \operatorname{tr}(\underline{K}^2) \right) \\ =& \frac{1}{4} \left((n-1) K_0^2 - K_a K_a \right) \\ =& \frac{1}{4} \left(n \delta_{\alpha 0} \delta_{\beta 0} - \delta_{\alpha\beta} \right) K_\alpha K_\beta, \\ s_3 =& \frac{1}{6} \left(\operatorname{tr}^3(\underline{K}) - 3 \operatorname{tr}(\underline{K}^2) \operatorname{tr}(\underline{K}) + 2 \operatorname{tr}(\underline{K}^3) \right). \end{aligned} \tag{2.21}$$

With the theorem 1 and (2.20) we have expressed the rank properties of the matrix $\underline{K}$ in terms of its eigenvalues, respectively, traces of powers of the matrix $\underline{K}$.

Based on theorem 1, (2.20), and (2.21), we can show that the gauge orbits of the n Higgs-boson doublet fields are in one to one correspondance to the vectors $(K_0, \ldots, K_{n^2-1})^{\mathrm{T}}$ in the n^2–dimensional space $\mathbb{R}_{n^2}$ satisfying

$$\begin{aligned} &s_1 \geq 0, s_2 \geq 0, \\ &s_3 = \ldots = s_n = 0. \end{aligned} \tag{2.22}$$

Here the s_k, $k = 1, \ldots, n$, are constructed from the matrix $\underline{K} = K_\alpha \lambda_\alpha / 2$ according to (2.20), (2.21). That is, to every gauge orbit of the Higgs-boson fields corresponds exactly one vector (K_α) satisfying (2.22) and vice versa. The first two relations of (2.22) are analogous to the *light cone* conditions of the THDM; see (36) of [2]. The remaining relations in the case $n > 2$ are specific for the nHDM.

Another way to parametrise all positive semidefinite matrices $\underline{K}$ of rank 1 and rank 2 is as follows.

For rank 1 the matrix $\underline{K}$ has only one eigenvalue unequal zero, say $\kappa_1 > 0$, $\kappa_2 = \ldots = \kappa_n = 0$. Let $\boldsymbol{w}$ be a normalised eigenvector of $\underline{K}$ to κ_1. Then we have

$$\begin{aligned} &\underline{K} = \sqrt{\frac{n}{2}} K_o \boldsymbol{w} \boldsymbol{w}^\dagger, \\ &\boldsymbol{w}^\dagger \boldsymbol{w} = 1, \\ &K_0 > 0, \quad \kappa_1 = \sqrt{\frac{n}{2}} K_0. \end{aligned} \tag{2.23}$$

For the bilinears we get from (2.23)

$$K_\alpha = \operatorname{tr}(\underline{K} \lambda_\alpha) = \sqrt{\frac{n}{2}} K_0 \boldsymbol{w}^\dagger \lambda_\alpha \boldsymbol{w}. \tag{2.24}$$

Clearly, for any normalised vector $\boldsymbol{w}$ from $\mathbb{C}_n$ we get with (2.23) a positive semidefinite matrix $\underline{K}$ of rank 1.

For rank 2 the matrix $\underline{K}$ has exactly two eigenvalues larger than zero. Without loss of generality we can set

$$\begin{aligned} &\kappa_1 = \sqrt{\frac{n}{2}} K_0 \sin^2(\chi), \\ &\kappa_2 = \sqrt{\frac{n}{2}} K_0 \cos^2(\chi), \\ &K_0 > 0, \quad 0 < \chi \leq \frac{\pi}{4}. \end{aligned} \tag{2.25}$$

Let $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$ be orthonormal eigenvectors of $\underline{K}$ to κ_1 and κ_2, respectively. We have then

$$\begin{aligned} &\underline{K} = \sqrt{\frac{n}{2}} K_0 \Big(\sin^2(\chi) \boldsymbol{w}_1 \boldsymbol{w}_1^\dagger + \cos^2(\chi) \boldsymbol{w}_2 \boldsymbol{w}_2^\dagger \Big), \\ &\boldsymbol{w}_i^\dagger \boldsymbol{w}_j = \delta_{ij}, \\ &K_0 > 0, \quad 0 < \chi \leq \frac{\pi}{4}. \end{aligned} \tag{2.26}$$

Here we get for the bilinears

$$\begin{aligned} K_\alpha = \operatorname{tr}(\underline{K} \lambda_\alpha) = \sqrt{\frac{n}{2}} K_0 \Big(&\sin^2(\chi), \boldsymbol{w}_1^\dagger \lambda_\alpha \boldsymbol{w}_1 \\ &+ \cos^2(\chi), \boldsymbol{w}_2^\dagger \lambda_\alpha \boldsymbol{w}_2 \Big). \end{aligned} \tag{2.27}$$

Clearly, the reverse also holds. For any two orthonormal vectors $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$ the construction (2.26) gives a positive semidefinite matrix $\underline{K}$ of rank 2.

With (2.23) and (2.26) we have the general parametrisation of all positive semidefinite matrices of rank 1 and rank 2, respectively. The parametrisations of the corresponding bilinears are given in (2.24) and (2.27), respectively. Based on the bilinears we shall in the following discuss the potential, basis transformations, stability, minimization, and electroweak symmetry breaking of the general nHDM.

3. THE nHDM POTENTIAL AND BASIS TRANSFORMATIONS

We now write the nHDM potential in terms of the bilinear coefficients, K_0, K_a, $a = 1, \ldots, n^2 - 1$,

$$V = \xi_0 K_0 + \xi_a K_a + \eta_{00} K_0^2 + 2K_0 \eta_a K_a + K_a \eta_{ab} K_b, \tag{3.1}$$

where the $n^2(n^2 + 3)/2$ parameters ξ_0, ξ_a, η_{00}, η_a and $\eta_{ab} = \eta_{ba}$ are real. The potential (3.1) consists of all possible linear and quadratic terms of the bilinears, corresponding to quadratic and quartic terms of the Higgs-boson doublets. Terms of higher order should not appear in the potential with view of renormalizability. Moreover, any constant term in the potential can be dropped and therefore (3.1) is the most general nHDM potential. We introduce the notation

$$\begin{aligned} \boldsymbol{K} &= (K_1, \ldots, K_{n^2-1})^{\mathrm{T}}, \quad \boldsymbol{\xi} = (\xi_1, \ldots, \xi_{n^2-1})^{\mathrm{T}}, \\ \boldsymbol{\eta} &= (\eta_1, \ldots, \eta_{n^2-1})^{\mathrm{T}}, \quad E = (\eta_{ab}), \\ \left(\tilde{E}_{\alpha\beta}\right) &= \begin{pmatrix} \eta_{00} & \eta_b \\ \eta_a & \eta_{ab} \end{pmatrix} = \begin{pmatrix} \eta_{00} & \boldsymbol{\eta}^{\mathrm{T}} \\ \boldsymbol{\eta} & E \end{pmatrix}. \end{aligned} \tag{3.2}$$

With this we can write the potential (3.1) as

$$V = \xi_\alpha K_\alpha + K_\alpha \tilde{E}_{\alpha\beta} K_\beta. \tag{3.3}$$

Now we consider a change of basis of the Higgs-boson fields, $\varphi_i(x) \to \varphi'_i(x)$, with

$$\begin{pmatrix} \varphi'_1(x)^{\mathrm{T}} \\ \vdots \\ \varphi'_n(x)^{\mathrm{T}} \end{pmatrix} = U \begin{pmatrix} \varphi_1(x)^{\mathrm{T}} \\ \vdots \\ \varphi_n(x)^{\mathrm{T}} \end{pmatrix}, \tag{3.4}$$

where $U \in U(n)$ is a $n \times n$ unitary transformation, that is, $U^\dagger U = \mathbb{1}_n$. From (3.4) we find $\phi'(x) = U\phi(x)$, and for the matrix $\underline{K}$ (2.4) and the bilinears

$$\underline{K}'(x) = U \underline{K}(x) U^\dagger, \tag{3.5}$$

$$K'_0(x) = K_0(x), \qquad K'_a(x) = R_{ab}(U) K_b(x). \tag{3.6}$$

Here $R_{ab}(U)$ is defined by

$$U^\dagger \lambda_a U = R_{ab}(U)\, \lambda_b. \tag{3.7}$$

The $(n^2 - 1) \times (n^2 - 1)$ matrix $R(U)$ has the properties

$$R^*(U) = R(U), \quad R^{\mathrm{T}}(U)\, R(U) = \mathbb{1}_{n^2-1},\ \det(R(U)) = 1, \tag{3.8}$$

that is, $R(U) \in SO(n^2 - 1)$. Let us note that the $R(U)$ form only a subset of $SO(n^2 - 1)$.

A pure phase transformation, $U = \exp(i\alpha)\mathbb{1}_n$, plays no role for the bilinears. We will, therefore, consider here only transformations (3.4) with $U \in SU(n)$. In the transformation of the bilinears (3.6) $R_{ab}(U)$ is then the $(n^2 - 1) \times (n^2 - 1)$ matrix corresponding to U in the adjoint representation of $SU(n)$.

Under the replacement (3.6), the Higgs potential (3.1) remains unchanged if we perform an appropriate simultaneous transformation of the parameters

$$\begin{aligned} \xi'_0 &= \xi_0, & \boldsymbol{\xi}' &= R(U)\, \boldsymbol{\xi}, \\ \eta'_{00} &= \eta_{00}, & \boldsymbol{\eta}' &= R(U)\, \boldsymbol{\eta}, \\ E' &= R(U)\, E\, R^{\mathrm{T}}(U). \end{aligned} \tag{3.9}$$

A realistic n-Higgs-doublet model contains besides the Higgs potential kinetic terms for the Higgs-boson doublets as well as Yukawa couplings which couple the Higgs-boson doublets to fermions.

Under a basis transformation, that is, a transformation (3.4) of the Higgs-boson doublets, or in terms of the bilinears, a transformation (3.6), the kinetic terms of the Higgs doublets are kept invariant. But in general the Yukawa couplings are *not* invariant under such a change of basis.

4. STABILITY OF THE nHDM

Now we study stability of the general nHDM potential (3.1), given in terms of the bilinears K_0 and $\boldsymbol{K}$ on the domain determined by (2.22). This is done in an analogous way to the cases with $n = 2, 3$, that is the THDM and the 3HDM; see [2, 3]. The case $\sqrt{n/2}K_0 = \varphi_1^\dagger \varphi_1 + \ldots + \varphi_n^\dagger \varphi_n = 0$ corresponds to vanishing Higgs-boson fields and $V = 0$. For $K_0 > 0$ we define

$$\underline{k} = \frac{\underline{K}}{K_0}, \quad k_\alpha = \frac{K_\alpha}{K_0}, \quad \boldsymbol{k} = (k_1, \ldots, k_{n^2-1})^{\mathrm{T}}. \tag{4.1}$$

Now we write the rank conditions of theorem 1 in terms of $\boldsymbol{k}$. With help of (2.8) we see that $\underline{K} = K_0 \cdot (\lambda_0 + k_a \lambda_a)/2$. Therefore, the expressions s_k (2.20) are proportional to K_0^k. We define the dimensionless expressions $\bar{s}_k$ by

$$\bar{s}_k = \frac{s_k}{K_0^k}, \tag{4.2}$$

and get from (2.20)

$$\begin{aligned} \bar{s}_0 =& 1, \\ \bar{s}_1 =& \frac{1}{2} \operatorname{tr}(\lambda_0 + k_a \lambda_a) = \sqrt{\frac{n}{2}}, \\ \bar{s}_2 =& \frac{1}{2} \left(\frac{1}{2} \operatorname{tr}^2(\lambda_0 + k_a \lambda_a) - \frac{1}{4} \operatorname{tr}([\lambda_0 + k_a \lambda_a]^2) \right) = \\ & \frac{1}{4} \left(n - 1 - k_a k_a \right), \\ \bar{s}_k =& \frac{1}{k} \sum_{i=1}^{k} (-1)^{i-1} \bar{s}_{k-i} \operatorname{tr}\left(\left[\frac{\lambda_0 + k_a \lambda_a}{2} \right]^i \right), \ k = 1, \ldots, n\,. \end{aligned} \tag{4.3}$$

In terms of the k_a we have for $\boldsymbol{k}$ the domain $\mathcal{D}_{\boldsymbol{k}}$:

$$\begin{aligned} &\bar{s}_2 \geq 0, \\ &\bar{s}_3 = \bar{s}_4 = \ldots = \bar{s}_n = 0. \end{aligned} \tag{4.4}$$

The domain boundary, $\partial\mathcal{D}_{\boldsymbol{k}}$, is given by

$$\bar{s}_2 = \frac{1}{4}\,(n-1-k_a k_a) = 0. \tag{4.5}$$

From (3.1) and (4.1) we obtain, for $K_0 > 0$, $V = V_2 + V_4$ with

$$V_2 = K_0\, J_2(\boldsymbol{k}), \quad J_2(\boldsymbol{k}) := \xi_0 + \boldsymbol{\xi}^{\mathrm{T}}\boldsymbol{k}, \tag{4.6}$$
$$V_4 = K_0^2\, J_4(\boldsymbol{k}), \quad J_4(\boldsymbol{k}) := \eta_{00} + 2\boldsymbol{\eta}^{\mathrm{T}}\boldsymbol{k} + \boldsymbol{k}^{\mathrm{T}}E\boldsymbol{k} \tag{4.7}$$

where we introduce the functions $J_2(\boldsymbol{k})$ and $J_4(\boldsymbol{k})$ on the domain (4.4).

Stability of the potential means that it is bounded from below. The stability follows from the behaviour of V in the limit $K_0 \to \infty$, hence, by the signs of $J_4(\boldsymbol{k})$ and $J_2(\boldsymbol{k})$ in (4.6), (4.7). For a model to be at least *marginally* stable, the conditions

$$\begin{aligned} &J_4(\boldsymbol{k}) > 0 \quad \text{or} \\ &J_4(\boldsymbol{k}) = 0 \quad \text{and} \quad J_2(\boldsymbol{k}) \geq 0 \end{aligned} \tag{4.8}$$

for all $\boldsymbol{k} \in \mathcal{D}_{\boldsymbol{k}}$, that is, all $\boldsymbol{k}$ satisfying (4.4) are necessary and sufficient, since this is equivalent to $V \geq 0$ for $K_0 \to \infty$ in all possible allowed directions $\boldsymbol{k}$. The more strict stability property $V \to \infty$ for $K_0 \to \infty$ and any allowed $\boldsymbol{k}$ requires V to be stable either in the strong or the weak sense. For strong stability we require

$$J_4(\boldsymbol{k}) > 0 \tag{4.9}$$

for all $\boldsymbol{k} \in \mathcal{D}_{\boldsymbol{k}}$; see (4.4). For stability in the weak sense we require for all $\boldsymbol{k} \in \mathcal{D}_{\boldsymbol{k}}$

$$\begin{aligned} &J_4(\boldsymbol{k}) \geq 0, \\ &J_2(\boldsymbol{k}) > 0 \text{ for all } \boldsymbol{k} \text{ where } J_4(\boldsymbol{k}) = 0. \end{aligned} \tag{4.10}$$

In order to check that $J_4(\boldsymbol{k})$ is positive (semi-)definite, it is sufficient to consider its value for all stationary points on the domain $\mathcal{D}_{\boldsymbol{k}}$. This is true because the global minimum of the continuous function $J_4(\boldsymbol{k})$ is reached on the compact domain $\mathcal{D}_{\boldsymbol{k}}$, and since the global minimum is among the stationary points.

In order to find the stationary points of $J_4(\boldsymbol{k})$ in the interior of the domain $\mathcal{D}_{\boldsymbol{k}}$ we note that here $\underline{k}$ of (4.1) has rank 2. Therefore, $\underline{k}$ and k_α can be represented as shown in (2.26) and (2.27), respectively, but setting $K_0 = 1$. This gives

$$k_\alpha = \sqrt{\frac{n}{2}}\Big(\sin^2(\chi)\boldsymbol{w}_1^\dagger\lambda_\alpha\boldsymbol{w}_1 + \cos^2(\chi)\boldsymbol{w}_2^\dagger\lambda_\alpha\boldsymbol{w}_2\Big) \tag{4.11}$$

where

$$\begin{aligned} &\boldsymbol{w}_1^\dagger\boldsymbol{w}_1 - 1 = 0, \\ &\boldsymbol{w}_2^\dagger\boldsymbol{w}_2 - 1 = 0, \\ &\frac{1}{2}(\boldsymbol{w}_1^\dagger\boldsymbol{w}_2 + \boldsymbol{w}_2^\dagger\boldsymbol{w}_1) = 0, \\ &\frac{1}{2i}(\boldsymbol{w}_1^\dagger\boldsymbol{w}_2 - \boldsymbol{w}_2^\dagger\boldsymbol{w}_1) = 0, \end{aligned} \tag{4.12}$$

$$0 < \chi \leq \frac{\pi}{4}. \tag{4.13}$$

We have to find the stationary points of

$$J_4(\boldsymbol{k}) = k_\alpha \tilde{E}_{\alpha\beta} k_\beta \tag{4.14}$$

under the constraints (4.12) and (4.13). The variation is with respect to the real and imaginary parts of the components of $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$ and to χ. It is easy to check that the gradient matrix of the four constraints (4.12) has rank 4. Therefore, we can use the Lagrange method and add these constraints with four multipliers to J_4 (4.14). We construct the function

$$\begin{aligned} &F(\boldsymbol{w}_1^\dagger, \boldsymbol{w}_1, \boldsymbol{w}_2^\dagger, \boldsymbol{w}_2, \chi, u_1, u_2, u_3, u_4) = \\ &J_4(\boldsymbol{k}) - u_1(\boldsymbol{w}_1^\dagger\boldsymbol{w}_1 - 1) - u_2(\boldsymbol{w}_2^\dagger\boldsymbol{w}_2 - 1) \\ &- u_3\frac{1}{2}(\boldsymbol{w}_1^\dagger\boldsymbol{w}_2 + \boldsymbol{w}_2^\dagger\boldsymbol{w}_1) - u_4\frac{1}{2i}(\boldsymbol{w}_1^\dagger\boldsymbol{w}_2 - \boldsymbol{w}_2^\dagger\boldsymbol{w}_1) \end{aligned} \tag{4.15}$$

where $\boldsymbol{k} = (k_a)$ is to be inserted from (4.11). The equations for the stationary points of J_4 are then obtained from

$$\begin{aligned} &\nabla_{\boldsymbol{w}_1^\dagger, \boldsymbol{w}_2^\dagger, \chi, u_1, u_2, u_3, u_4} F(\boldsymbol{w}_1^\dagger, \boldsymbol{w}_1, \boldsymbol{w}_2^\dagger, \boldsymbol{w}_2, \chi, u_1, u_2, u_3, u_4) = 0, \\ &\text{for } 0 < \chi < \frac{\pi}{4}. \end{aligned} \tag{4.16}$$

For the boundary value $\chi = \pi/4$ we have

$$\begin{aligned} &\nabla_{\boldsymbol{w}_1^\dagger, \boldsymbol{w}_2^\dagger, u_1, u_2, u_3, u_4} \\ &\qquad F(\boldsymbol{w}_1^\dagger, \boldsymbol{w}_1, \boldsymbol{w}_2^\dagger, \boldsymbol{w}_2, \pi/4, u_1, u_2, u_3, u_4) = 0. \end{aligned} \tag{4.17}$$

The gradients of F with respect to $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$ give the hermitian conjugate of the gradients with respect to $\boldsymbol{w}_1^\dagger$ $\boldsymbol{w}_2^\dagger$, respectively, in (4.16) and (4.17), thus, nothing new. For $\underline{k}$, (4.1), of rank 1 we use (2.23), (2.24) to get

$$\begin{aligned} \underline{k} &= \sqrt{\frac{n}{2}}\boldsymbol{w}\boldsymbol{w}^\dagger, \\ k_\alpha &= \sqrt{\frac{n}{2}}\boldsymbol{w}^\dagger\lambda_\alpha\boldsymbol{w} \end{aligned} \tag{4.18}$$

where we have the constraint

$$\boldsymbol{w}^\dagger\boldsymbol{w} - 1 = 0. \tag{4.19}$$

We easily check that the gradient matrix of the constraint has here rank 1. Therefore we add (4.19) with one Lagrange multiplier to J_4 and get

$$\begin{aligned} F(\boldsymbol{w}^\dagger, \boldsymbol{w}, u) &= J_4(\boldsymbol{k}) - u(\boldsymbol{w}^\dagger\boldsymbol{w} - 1) = \\ &\frac{n}{2}\boldsymbol{w}^\dagger\lambda_\alpha\boldsymbol{w}\tilde{E}_{\alpha\beta}\boldsymbol{w}^\dagger\lambda_\beta\boldsymbol{w} - u(\boldsymbol{w}^\dagger\boldsymbol{w} - 1). \end{aligned} \tag{4.20}$$

The equations determining the stationary points of $J_4(\boldsymbol{k})$ on the boundary $\partial\mathcal{D}_{\boldsymbol{k}}$, that is, for $\underline{k}$ of rank 1, are then

$$\nabla_{\boldsymbol{w}^\dagger, u} F(\boldsymbol{w}^\dagger, \boldsymbol{w}, u) = 0. \tag{4.21}$$

All stationary points obtained from (4.16), (4.17), and (4.21) have to fulfill the condition $J_4(\boldsymbol{k}) > 0$ for stability in the strong sense. If for all stationary points we have $J_4(\boldsymbol{k}) \geq 0$, then for every solution $\boldsymbol{k}$ with $J_4(\boldsymbol{k}) = 0$ we have to have $J_2(\boldsymbol{k}) > 0$ for stability in the weak sense, or at least $J_2(\boldsymbol{k}) = 0$ for *marginal* stability. If none of these conditions is fulfilled, that is, if we find at least one stationary direction $\boldsymbol{k}$ with $J_4(\boldsymbol{k}) < 0$ or $J_4(\boldsymbol{k}) = 0$ but $J_2(\boldsymbol{k}) < 0$, the potential is unstable.

Our discussion above of the stability conditions for the nHDM potential generalises the results for the THDM and the 3HDM in [2] and [3], respectively. We have been careful to use in our present paper a compatible notation. The stability properties of the general THDM and 3HDM potentials were analysed in detail in chapters 4 of [2] and [3], respectively. Also explicit examples of THDM and 3HDM potentials, using conventional parametrisations, were discussed in these references.

5. ELECTROWEAK SYMMETRY BREAKING IN THE nHDM

Now we assume that the nHDM potential is stable, that is, it is bounded from below. This means that the global minimum will be among the stationary points of V. We now want to distinguish the different cases of minima with respect to the underlying electroweak symmetry. We shall in the following present the corresponding stationarity equations.

We have seen in section 2, that the space of the Higgs-boson doublets is determined, up to electroweak gauge transformations, by the space of the hermitian $n \times n$ matrices $\underline{K}$ with rank smaller or equal 2. Based on the fact that the rank of the matrix $\underline{K}$ is equal to the rank of the Higgs-boson field matrix ϕ (2.3) we can distinguish the different types of minima with respect to electroweak symmetry breaking as follows. We start with writing at the global minimum, that is, the vacuum configuration, the $n \times 2$ matrix of the Higgs-boson fields as

$$\langle \phi \rangle = \begin{pmatrix} v_1^+ & v_1^0 \\ \vdots & \vdots \\ v_n^+ & v_n^0 \end{pmatrix}. \tag{5.1}$$

Suppose, this matrix has rank 2, then we cannot, by a $SU(2)_L \times U(1)_Y$ transformation, get a form with all charged components v_i^+, $i = 1, \ldots, n$ vanishing. Hence, the $SU(2)_L \times U(1)_Y$ group is fully broken. Next, suppose that at the global minimum the matrix $\langle \phi \rangle$ has rank one. Then we can, by a $SU(2)_L \times U(1)_Y$ transformation get a form with all charged components v_i^+ vanishing. Further, we can identity the unbroken $U(1)$ gauge group with the electromagnetic gauge group $U(1)_{em}$. Hence, a minimum with rank one corresponds to the electroweak-symmetry breaking $SU(2)_L \times U(1)_Y \ \to U(1)_{em}$. Eventually, suppose we get a vanishing matrix at the minimum, $\langle \phi \rangle = 0$. This corresponds to an unbroken electroweak symmetry. Let us note that only a minimum with a partially broken electroweak symmetry is physically acceptable.

We study now the matrix $\underline{K}_v$ corresponding to $\langle \phi \rangle$ (5.1)

$$\underline{K}_v = \langle \phi \rangle \langle \phi \rangle^\dagger = \frac{1}{2} K_{v\alpha} \lambda_\alpha. \tag{5.2}$$

For an acceptable vacuum $\langle \phi \rangle$, $\underline{K}_v$ must have rank 1. From theorem 1 we see that $\underline{K}_v$ has rank 1 and is positive semidefinite if and only if

$$\begin{aligned} \operatorname{tr} \underline{K}_v &= \sqrt{\frac{n}{2}} K_{v0} > 0, \\ \langle s_2 \rangle &= \ldots = \langle s_n \rangle = 0. \end{aligned} \tag{5.3}$$

We can bring the vacuum value $\langle \phi \rangle$ of rank 1, by suitable $SU(2)_L \times U(1)_Y$ and $U(n)$ transformations (3.4), to the form

$$\langle \phi \rangle = \begin{pmatrix} 0 & 0 \\ \vdots & \vdots \\ 0 & v_0/\sqrt{2} \end{pmatrix}, \qquad v_0 > 0. \tag{5.4}$$

In a realistic model v_0 must be the usual Higgs-boson vacuum expectation value,

$$v_0 \approx 246 \text{ GeV}. \tag{5.5}$$

With (5.4) we find in this basis a simple form for $\underline{K}_v$ respectively $K_{v\alpha}$:

$$\begin{aligned} \underline{K}_v &= \frac{1}{2} \begin{pmatrix} 0 & \ldots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & 0 & 0 \\ 0 & \cdots & 0 & v_0^2 \end{pmatrix} = \frac{1}{2} K_{v\alpha} \lambda_\alpha, \\ \left(K_{v\alpha} \right) &= \frac{1}{\sqrt{2n}} v_0^2 \left(1, 0, \ldots, 0, -\sqrt{n-1} \right)^{\mathrm{T}}. \end{aligned} \tag{5.6}$$

We note that another possible choice for the vacuum expectation value, achievable by suitable transformations from $SU(2)_L \times U(1)_Y$ and $U(n)$ (3.4), is

$$\langle \phi \rangle = \begin{pmatrix} 0 & v_0/\sqrt{2} \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix}, \qquad v_0 > 0. \tag{5.7}$$

Here we get

$$\underline{K}_v = \frac{1}{2} \begin{pmatrix} v_0^2 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}. \tag{5.8}$$

In the cases where $\langle \phi \rangle$ of (5.1) has rank 2 or rank 0 also the matrix $\underline{K}_v$, (5.2), has rank 2 or zero, respectively. The corresponding conditions for $\underline{K}_v$ are given explicitly in theorem 1 if we replace all expressions by their vacuum

expectation values, that is, $\underline{K}$ by $\underline{K}_v$, K_α by $K_{v\alpha}$ and s_i by $\langle s_i \rangle$. We summarise our findings for the vacuum expectation values to a given potential V as follows.

Suppose $\langle \phi \rangle$ is the vacuum expectation value of the Higgs-boson field matrix to a given, stable, potential V and $\underline{K}_v = \langle \phi \rangle \langle \phi \rangle^\dagger = K_{v\alpha}\lambda_\alpha/2$. The gauge symmetry $SU(2)_L \times U(1)_Y$ is fully broken by the vacuum if and only if

$$K_{v0} > 0, \qquad (n-1)K_{v0}^2 - K_{va}K_{va} > 0. \tag{5.9}$$

We have the breaking $SU(2)_L \times U(1)_Y \to U(1)_{em}$ if and only if

$$K_{v0} > 0, \qquad (n-1)K_{v0}^2 - K_{va}K_{va} = 0. \tag{5.10}$$

We have no breaking of $SU(2)_L \times U(1)_Y$ if and only if

$$K_{v\alpha} = 0. \tag{5.11}$$

Clearly, we have always

$$\langle s_3 \rangle = \cdots = \langle s_n \rangle = 0. \tag{5.12}$$

6. STATIONARY POINTS

Now suppose we have a stable potential. We shall study the stationarity equations with view on the electroweak symmetry breaking behavior. If the potential is stable, the global minimum is among the stationary points of V. We classify the stationary points by the rank of the stationarity matrix $\underline{K}$. We will apply the conditions for $\underline{K}$ having rank 0, 1, 2 as given in theorem 1 and (2.23) to (2.27).

Rank 0, that is, $\underline{K} = 0$, respectively $K_\alpha = 0$, $\alpha = 0, \ldots, n^2-1$, corresponds to a stationary point of V with value $V(K_\alpha) = 0$.

All stationarity matrices $\underline{K} = K_\alpha\lambda_\alpha/2$ of rank 1 are obtained from the following system of equations. We represent $\underline{K}$ of rank 1 according to (2.23). Then K_α is given by (2.24) and V (3.3) by

$$\begin{aligned} V(K_\alpha) &= \xi_\alpha K_\alpha + K_\alpha \tilde{E}_{\alpha\beta} K_\beta \\ &= \xi_\alpha \sqrt{\frac{n}{2}} K_0 \boldsymbol{w}^\dagger \lambda_\alpha \boldsymbol{w} \\ &\quad + \Big(\sqrt{\frac{n}{2}} K_0\Big)^2 (\boldsymbol{w}^\dagger \lambda_\alpha \boldsymbol{w}) \tilde{E}_{\alpha\beta} (\boldsymbol{w}^\dagger \lambda_\beta \boldsymbol{w}) \end{aligned} \tag{6.1}$$

where

$$K_0 > 0, \qquad \boldsymbol{w}^\dagger \boldsymbol{w} - 1 = 0. \tag{6.2}$$

Taking the constraint equation in (6.2) into account with a Lagrange multiplier u we get the following function to determine the stationary points of V with $\underline{K}$ of rank 1

$$F(\boldsymbol{w}^\dagger, \boldsymbol{w}, K_0, u) = V(K_\alpha) - u(\boldsymbol{w}^\dagger \boldsymbol{w} - 1). \tag{6.3}$$

The gradient matrix of the constraint has rank 1 as required and we get the equations

$$\begin{aligned} &\nabla_{\boldsymbol{w}^\dagger, K_0, u} F(\boldsymbol{w}^\dagger, \boldsymbol{w}, K_0, u) = 0, \\ &K_0 > 0. \end{aligned} \tag{6.4}$$

All stationarity matrices $\underline{K} = K_\alpha\lambda_\alpha/2$ of rank 2 are obtained from the following system of equations. We represent $\underline{K}$ of rank 2 and the corresponding K_α as in (2.26) and (2.27), respectively, and take the constraints for $\boldsymbol{w}_i$ into account with the help of four Lagrange multipliers; cf. (4.12), (4.15). We have then to determine the stationary points of the function

$$\begin{aligned} &F(\boldsymbol{w}_1^\dagger, \boldsymbol{w}_1, \boldsymbol{w}_2^\dagger, \boldsymbol{w}_2, \chi, K_0, u_1, u_2, u_3, u_4) = \\ &V(K_\alpha) - u_1(\boldsymbol{w}_1^\dagger \boldsymbol{w}_1 - 1) - u_2(\boldsymbol{w}_2^\dagger \boldsymbol{w}_2 - 1) \\ &- u_3 \frac{1}{2}(\boldsymbol{w}_1^\dagger \boldsymbol{w}_2 + \boldsymbol{w}_2^\dagger \boldsymbol{w}_1) - u_4 \frac{1}{2i}(\boldsymbol{w}_1^\dagger \boldsymbol{w}_2 - \boldsymbol{w}_2^\dagger \boldsymbol{w}_1). \end{aligned} \tag{6.5}$$

The stationarity equations are then

$$\begin{aligned} &\nabla_{\boldsymbol{w}_1^\dagger, \boldsymbol{w}_2^\dagger, \chi, K_0, u_1, u_2, u_3, u_4} \\ &\qquad F(\boldsymbol{w}_1^\dagger, \boldsymbol{w}_1, \boldsymbol{w}_2^\dagger, \boldsymbol{w}_2, \chi, K_0, u_1, u_2, u_3, u_4) = 0, \\ &\qquad\qquad 0 < \chi < \frac{\pi}{4}, \quad K_0 > 0. \end{aligned} \tag{6.6}$$

For $\chi = \pi/4$ we get

$$\begin{aligned} &\nabla_{\boldsymbol{w}_1^\dagger, \boldsymbol{w}_2^\dagger, K_0, u_1, u_2, u_3, u_4} \\ &\qquad F(\boldsymbol{w}_1^\dagger, \boldsymbol{w}_1, \boldsymbol{w}_2^\dagger, \boldsymbol{w}_2, \pi/4, K_0, u_1, u_2, u_3, u_4) = 0, \\ &\qquad\qquad K_0 > 0. \end{aligned} \tag{6.7}$$

The stationarity matrix $\underline{K} = K_\alpha\lambda_\alpha/2$ with the lowest value of $V(K_0, \ldots, K_{n^2-1})$ gives the global-minimum matrix $\underline{K}_v$ of the potential. In general there may be degenerate global minima with the same potential value. It was shown that systems of equations of the type (6.4), (6.6), and (6.7) can be solved via the Groebner-basis approach or homotopy continuation; see for instance [25, 26].

7. THE POTENTIAL AFTER SYMMETRY BREAKING

In this section we present the calculation of the physical Higgs-boson masses in the nHDM. Suppose that the potential is stable and leads to the desired electroweak symmetry breaking, that is, $\underline{K}_v$ has rank 1. From the previous discussion follows that the global minimum has then to be obtained from a solution of the set of equations (6.4).

Using (6.1) we can write (6.4) explicitly as follows

$$\begin{aligned} &\sqrt{\frac{n}{2}} K_0 \Big[\xi_\alpha + 2\tilde{E}_{\alpha\beta} \sqrt{\frac{n}{2}} K_0 (\boldsymbol{w}^\dagger \lambda_\beta \boldsymbol{w}) \Big] \lambda_\alpha \boldsymbol{w} \\ &\qquad - u\boldsymbol{w} = 0, \end{aligned} \tag{7.1}$$

$$\boldsymbol{w}^\dagger\boldsymbol{w} - 1 = 0, \tag{7.2}$$

$$\left[\xi_\alpha + 2\tilde{E}_{\alpha\beta}\sqrt{\frac{n}{2}}K_0(\boldsymbol{w}^\dagger\lambda_\beta\boldsymbol{w})\right](\boldsymbol{w}^\dagger\lambda_\alpha\boldsymbol{w}) = 0, \tag{7.3}$$

$$K_0 > 0. \tag{7.4}$$

Multiplying (7.1) with $\boldsymbol{w}^\dagger$ from left and using (7.2) and (7.3) we find

$$u = 0. \tag{7.5}$$

The vacuum value $\underline{K}_v$ is solution of this system of equations. In the following we will always work in a basis where $\langle\phi\rangle$ and $\underline{K}_v$ have the forms (5.4) and (5.6), respectively. Furthermore, it is convenient to use instead of $\alpha = 0, 1, \ldots, n^2-2, n^2-1$ the basis $+, 1, \ldots, n^2-2, -$; see appendix A. Thus, all indices ρ, σ, ... run over this latter index set in the following. From (5.6) we find

$$\begin{aligned}
\underline{K}_v &= \frac{1}{2}v_0^2\boldsymbol{e}_n\boldsymbol{e}_n^\dagger = \frac{1}{2\sqrt{2}}v_0^2\lambda_-,\\
(K_{v\rho}) &= \left(0,\ \ldots,\ 0,\ \tfrac{1}{\sqrt{2}}v_0^2\right),\\
K_{v-} &= \frac{1}{\sqrt{2}}v_0^2.
\end{aligned} \tag{7.6}$$

Here and in the following $\boldsymbol{e}_l$, $l \in \{1, \ldots, n\}$, are the usual Cartesian unit vectors in $\mathbb{C}_n$. We get now that for the solution vector $\boldsymbol{w}$ in (7.1) we have

$$\boldsymbol{w} = \boldsymbol{e}_n \tag{7.7}$$

and that

$$\sqrt{\frac{n}{2}}K_0\boldsymbol{w}^\dagger\lambda_\rho\boldsymbol{w} = K_{v\rho}. \tag{7.8}$$

We define

$$\begin{aligned}
\zeta_\rho &= \xi_\rho + 2\tilde{E}_{\rho\sigma}K_{v\sigma}\\
&= \xi_\rho + 2\tilde{E}_{\rho-}K_{v-}.
\end{aligned} \tag{7.9}$$

With this we can write (7.1), using $K_0 > 0$, in the basis $+, 1, \ldots, n^2-2, -$ as

$$\zeta_\rho\lambda_\rho\boldsymbol{e}_n = 0. \tag{7.10}$$

From the explicit construction and numbering scheme of the matrices λ_ρ in appendix A we see that we have

$$\begin{aligned}
&\lambda_\rho\boldsymbol{e}_n = 0, \qquad \text{for } \rho = +, 1, \ldots, (n-1)^2-1,\\
&\lambda_{(n-1)^2}\boldsymbol{e}_n = \boldsymbol{e}_1,\\
&\lambda_{(n-1)^2+1}\boldsymbol{e}_n = -i\boldsymbol{e}_1,\\
&\quad\vdots\\
&\lambda_{n^2-3}\boldsymbol{e}_n = \boldsymbol{e}_{n-1},\\
&\lambda_{n^2-2}\boldsymbol{e}_n = -i\boldsymbol{e}_{n-1},\\
&\lambda_-\boldsymbol{e}_n = \sqrt{2}\boldsymbol{e}_n.
\end{aligned} \tag{7.11}$$

Therefore, (7.10) gives

$$\begin{aligned}
&\big(\zeta_{(n-1)^2} - i\zeta_{(n-1)^2+1}\big)\boldsymbol{e}_1 + \ldots +\\
&\qquad\big(\zeta_{n^2-3} - i\zeta_{n^2-2}\big)\boldsymbol{e}_{n-1}\\
&\qquad\qquad + \zeta_-\sqrt{2}\boldsymbol{e}_n = 0.
\end{aligned} \tag{7.12}$$

Since all ζ_ρ are real we get as a result of the stationarity equations for $\underline{K}$ of rank 1 in our basis

$$\begin{aligned}
&\zeta_\rho = 0\\
&\text{for } \rho = (n-1)^2, (n-1)^2+1, \ldots, (n^2-2), -.
\end{aligned} \tag{7.13}$$

Now we turn to the Higgs-field matrix ϕ. As stated above we work in the basis where, in the unitary gauge, the vacuum-expectation value $\langle\phi\rangle$ has the form (5.4). For the original Higgs fields expressed in terms of the physical fields we get then

$$\begin{aligned}
\varphi_i(x) &= \begin{pmatrix} H_i^+(x)\\ \frac{1}{\sqrt{2}}\left(H_i^0(x) + iA_i^0(x)\right)\end{pmatrix}, \qquad i = 1, \ldots, n-1\\
\varphi_n(x) &= \frac{1}{\sqrt{2}}\begin{pmatrix} 0\\ v_0 + h_0(x)\end{pmatrix},
\end{aligned} \tag{7.14}$$

with v_0 real and positive, neutral fields $h_0(x)$, $H_i^0(x)$, $A_i^0(x)$, as well as the complex charged fields $H_i^+(x)$ with $i = 1, \ldots, n-1$. The negatively charged Higgs-boson fields are defined by $H_i^-(x) = \left(H_i^+(x)\right)^\dagger$. Hence, we get in the nHDM the physical fields

$$\begin{aligned}
&2n-1 \text{ neutral fields:}\quad H_i^0(x), A_i^0(x), h_0(x),\\
&n-1 \text{ charged fields:}\quad H_i^+(x),
\end{aligned} \tag{7.15}$$

with $i = 1, \ldots, n-1$. It is clear that the n original complex doublets of any nHDM, corresponding to $4n$ real degrees of freedom, yield $2n-1$ real fields and $n-1$ complex fields, with the 3 remaining degrees of freedom absorbed via the mechanism of electroweak symmetry breaking. Expressing the bilinears in the parametrization (7.14) via (2.4) and (2.8) we can write the potential in terms of the physical fields (7.15). We start by expanding all quantities in powers of the physical fields. This gives for the field matrix

$$\begin{aligned}
\phi(x) &= \langle\phi\rangle + \phi^{(1)}(x),\\
\phi^{(1)}(x) &= \begin{pmatrix}
H_1^+(x) & \frac{1}{\sqrt{2}}\left(H_1^0(x) + iA_1^0(x)\right)\\
\vdots & \vdots\\
H_{n-1}^+(x) & \frac{1}{\sqrt{2}}\left(H_{n-1}^0(x) + iA_{n-1}^0(x)\right)\\
0 & \frac{1}{\sqrt{2}}h_0(x)
\end{pmatrix}.
\end{aligned} \tag{7.16}$$

For $\underline{K}(x)$ we get

$$\begin{aligned}
\underline{K}(x) &= \underline{K}_v + \underline{K}^{(1)}(x) + \underline{K}^{(2)}(x),\\
\underline{K}^{(1)}(x) &= \phi^{(1)}(x)\langle\phi\rangle^\dagger + \langle\phi\rangle\phi^{(1)\dagger}(x),\\
\underline{K}^{(2)}(x) &= \phi^{(1)}(x)\phi^{(1)\dagger}(x).
\end{aligned} \tag{7.17}$$

Explicitly we get for $K_+^{(1)}(x)$

$$\begin{aligned} K_+^{(1)}(x) &= \mathrm{tr}\left(\lambda_+ \underline{K}^{(1)}(x)\right) \\ &= \mathrm{tr}\left(\langle\phi\rangle^\dagger \lambda_+ \phi^{(1)}(x) + h.c.\right) \\ &= 0, \end{aligned} \tag{7.18}$$

Similarly we show that the only non-zero components of $K_\rho^{(1)}(x)$ are as follows:

$$\begin{aligned} K_\rho^{(1)}(x) &= v_0 H_l^0(x), \\ &\qquad \text{for } \rho = (n-1)^2 + 2l - 2, \\ K_\rho^{(1)}(x) &= -\, v_0 A_l^0(x), \\ &\qquad \text{for } \rho = (n-1)^2 + 2l - 1, \\ K_-^{(1)}(x) &= v_0 \sqrt{2} h_0(x) \end{aligned} \tag{7.19}$$

where $l = 1, \dots, n-1$.

For the potential we write

$$V = V^{(0)} + V^{(1)} + V^{(2)} + V^{(3)} + V^{(4)} \tag{7.20}$$

and with (3.3) and (7.17) we get

$$\begin{aligned} V^{(0)} =& K_{v\rho}\xi_\rho + K_{v\rho}\tilde{E}_{\rho\sigma}K_{v\sigma}, \\ V^{(1)} =& K_\rho^{(1)}(x)\xi_\rho + 2K_\rho^{(1)}(x)\tilde{E}_{\rho\sigma}K_{v\sigma}, \\ V^{(2)} =& K_\rho^{(2)}(x)\xi_\rho + 2K_\rho^{(2)}(x)\tilde{E}_{\rho\sigma}K_{v\sigma} \\ &+ K_\rho^{(1)}(x)\tilde{E}_{\rho\sigma}K_\sigma^{(1)}(x), \\ V^{(3)} =& 2K_\rho^{(2)}(x)\tilde{E}_{\rho\sigma}K_\sigma^{(1)}(x), \\ V^{(4)} =& K_\rho^{(2)}(x)\tilde{E}_{\rho\sigma}K_\sigma^{(2)}(x). \end{aligned} \tag{7.21}$$

We shall now simplify the expressions for $V^{(0)}$, $V^{(1)}$, and $V^{(2)}$ using (7.6), (7.13), and (7.19). Writing $V^{(0)}$ as

$$\begin{aligned} V^{(0)} &= \frac{1}{2}K_{v\rho}\xi_\rho + \frac{1}{2}K_{v\rho}\left[\xi_\rho + 2\tilde{E}_{\rho\sigma}K_{v\sigma}\right] \\ &= \frac{1}{2}K_{v\rho}\left(\xi_\rho + \zeta_\rho\right) \end{aligned} \tag{7.22}$$

we find with (7.6) and (7.13)

$$\begin{aligned} V^{(0)} = \frac{1}{2}K_{v\rho}\xi_\rho &= \frac{1}{2}K_{v-}\xi_- \\ &= \frac{1}{2}v_0^2\frac{1}{\sqrt{2n}}\left(\xi_0 - \sqrt{n-1}\xi_{n^2-1}\right) \end{aligned} \tag{7.23}$$

which is the potential value at the vacuum. Next we consider $V^{(1)}$. With (7.9), (7.13), and (7.19) we get

$$\begin{aligned} V^{(1)} =& K_\rho^{(1)}(x)\left[\xi_\rho + 2\tilde{E}_{\rho\sigma}K_{v\sigma}\right] \\ =& K_\rho^{(1)}(x)\zeta_\rho = 0 \end{aligned} \tag{7.24}$$

since for each term in the above sum either $K_\rho^{(1)}(x) = 0$ or $\zeta_\rho = 0$, $\rho = +, 1, \dots, n^2-2, -$. This result must come out since we are expanding around the true minimum of the potential.

Finally we consider $V^{(2)}$. Using again (7.9), (7.13), (7.17), and (7.21) we can write this as

$$\begin{aligned} V^{(2)} =& K_\rho^{(2)}(x)\zeta_\rho + K_\rho^{(1)}(x)\tilde{E}_{\rho\sigma}K_\sigma^{(1)}(x) \\ =& \mathrm{tr}\left(\phi^{(1)\dagger}(x)\zeta_\rho\lambda_\rho\phi^{(1)}(x)\right) \\ &+ K_\rho^{(1)}(x)\tilde{E}_{\rho\sigma}K_\sigma^{(1)}(x). \end{aligned} \tag{7.25}$$

Here in the first term the sum runs only over $\rho = +, 1, \dots, (n-1)^2-1$ due to (7.13), in the second term only over $\rho, \sigma = (n-1)^2, \dots, n^2-2, -$ due to (7.18) and (7.19). The evaluation of these terms is straightforward. We define the fields

$$\begin{aligned} \mathcal{H}^+(x) &= \left(H_1^+(x), \dots, H_{n-1}^+(x)\right)^{\mathrm{T}}, \\ \mathcal{H}^0(x) &= \left(H_1^0(x), \dots, H_{n-1}^0(x)\right)^{\mathrm{T}}, \\ \mathcal{A}^0(x) &= \left(A_1^0(x), \dots, A_{n-1}^0(x)\right)^{\mathrm{T}}. \end{aligned} \tag{7.26}$$

Furthermore, we define $(n-1)\times(n-1)$ matrices

$$\mathscr{M}_{\mathrm{ch}}^2 = \left(\zeta_+\lambda_+ + \sum_{\rho=1}^{(n-1)^2-1}\zeta_\rho\lambda_\rho\right)\Bigg|_{\substack{\text{restricted to}\\ \text{the first } n-1 \text{ dimensions}}}, \tag{7.27}$$

$$\begin{aligned} \mathcal{E}_{HH} &= (2v_0^2\tilde{E}_{(n-1)^2+2l-2,(n-1)^2+2l'-2}), \\ \mathcal{E}_{HA} &= (-2v_0^2\tilde{E}_{(n-1)^2+2l-2,(n-1)^2+2l'-1}), \\ \mathcal{E}_{AH} &= (-2v_0^2\tilde{E}_{(n-1)^2+2l-1,(n-1)^2+2l'-2}), \\ \mathcal{E}_{AA} &= (2v_0^2\tilde{E}_{(n-1)^2+2l-1,(n-1)^2+2l'-1}) \\ &\text{where} \\ &l, l' \in \{1, \dots, n-1\}. \end{aligned} \tag{7.28}$$

We also need the $(n-1)\times 1$ matrices

$$\begin{aligned} \mathcal{E}_{H-} &= (2\sqrt{2}v_0^2\tilde{E}_{(n-1)^2+2l-2,-}) \\ &= (-2\xi_{(n-1)^2+2l-2}), \\ \mathcal{E}_{A-} &= (-2\sqrt{2}v_0^2\tilde{E}_{(n-1)^2+2l-1,-}) \\ &= (2\xi_{(n-1)^2+2l-1}), \end{aligned} \tag{7.29}$$

the $1\times(n-1)$ matrices

$$\begin{aligned} \mathcal{E}_{-H} &= (2\sqrt{2}v_0^2\tilde{E}_{-,(n-1)^2+2l'-2}) \\ &= (-2\xi_{(n-1)^2+2l'-2}), \\ \mathcal{E}_{-A} &= (-2\sqrt{2}v_0^2\tilde{E}_{-,(n-1)^2+2l'-1}) \\ &= (2\xi_{(n-1)^2+2l'-1}) \end{aligned} \tag{7.30}$$

and the scalar

$$\begin{aligned} \mathcal{E}_{--} &= 4v_0^2\tilde{E}_{--} = -2\sqrt{2}\xi_- \\ &= -\frac{4}{\sqrt{2n}}\left(\xi_0 - \sqrt{n-1}\xi_{n^2-1}\right) \\ &= -\frac{8}{v_0^2}V^{(0)}. \end{aligned} \tag{7.31}$$

In (7.29) to (7.31) we used (7.13) and (7.23). With all this we obtain for $V^{(2)}$

$$V^{(2)} = \mathcal{H}^{+\dagger}(x)\, \mathscr{M}^2_{ch}\, \mathcal{H}^{+}(x) \\ + \left(\mathcal{H}^{0\mathrm{T}}(x),\ \mathcal{A}^{0\mathrm{T}}(x),\ h_0(x)\right) \frac{1}{2} \mathscr{M}^2_{\mathrm{n}} \begin{pmatrix} \mathcal{H}^0(x), \\ \mathcal{A}^0(x), \\ h_0(x) \end{pmatrix} \tag{7.32}$$

where the mass matrix squared of the charged fields, $\mathscr{M}^2_{ch}$, is given in (7.27) and that of the neutral fields, $\mathscr{M}^2_{\mathrm{n}}$, is given by

$$\mathscr{M}^2_{\mathrm{n}} = \begin{pmatrix} \mathrm{Re}(\mathscr{M}^2_{ch}) + \mathcal{E}_{HH} & -\,\mathrm{Im}(\mathscr{M}^2_{ch}) + \mathcal{E}_{HA} & \mathcal{E}_{H-} \\ \mathrm{Im}(\mathscr{M}^2_{ch}) + \mathcal{E}_{AH} & \mathrm{Re}(\mathscr{M}^2_{ch}) + \mathcal{E}_{AA} & \mathcal{E}_{A-} \\ \mathcal{E}_{-H} & \mathcal{E}_{-A} & \mathcal{E}_{--} \end{pmatrix}. \tag{7.33}$$

Since we have assumed that we are dealing with the true vacuum, $V^{(0)}$ (7.23) must be below or at most equal to the potential value at $\underline{K} = 0$. That is, we must have

$$V^{(0)} \leq 0 \tag{7.34}$$

which implies, from (7.23),

$$\xi_0 - \sqrt{n-1}\,\xi_{n^2-1} \leq 0. \tag{7.35}$$

Usually the true vacuum is required to be below the value $V = 0$ corresponding to the trivial stationary point $K_\alpha = 0$ and then the strict inequalities must hold in (7.34) and (7.35). For the true vacuum the squared mass matrices of the physical Higgs bosons must be positive semidefinite:

$$\mathscr{M}^2_{ch} \geq 0, \quad \mathscr{M}^2_{\mathrm{n}} \geq 0. \tag{7.36}$$

Looking at $\mathscr{M}^2_{ch}$ we see that in general it will not lead to mass degeneracy of all charged physical Higgs bosons. This is confirmed by the study of simple examples [3]. For the case that we have

$$\mathcal{E}_{H-} = \mathcal{E}_{A-} = 0 \tag{7.37}$$

the field $h_0(x)$ is a mass eigenfield with mass squared value, see (7.31),

$$m^2_{h_0} = \mathcal{E}_{--} = -\frac{8}{v_0^2} V^{(0)}. \tag{7.38}$$

In this case the field $h_0(x)$ is called *aligned* with the vacuum expectation value.

8. CONCLUSION

The n-Higgs-doublet model has been studied as a generalization of the THDM and the 3HDM. Stability, electroweak symmetry breaking, and the stationary points of the Higgs potential have been discussed. We have presented explicit sets of equations allowing to determine the stability of any nHDM. In case of stability, the equations to determine the stationary points of the potential have been presented.

Of course, there are three types of vacuum solutions: with complete breaking, with no breaking, and with partial breaking of $SU(2)_L \times U(1)_Y$. For the latter case – the only one of physical interest – we have investigated the potential after symmetry breaking. The mass squared of the physical Higgs bosons have been given explicitly. For all these investigations we have found the use of the gauge-invariant bilinears very convenient. For numerical investigations of the stability and stationarity equations one has to solve polynomial equations. For this there are approaches available, like Groebner-bases or homotopy continuation, which are capable to solve these sets of equations. We have found that the degree of these polynomial equations is independent of the number n of Higgs bosons; see (4.15) to (4.17), (4.20), (4.21), (6.3) to (6.7), and (7.1) to (7.3). But the number of variables increases, in essence proportional to n. To conclude: we find it remarkable that, using the method of bilinears, one can get a rather good overview of the properties of the potentials of the nHDM, even if at first sight these potentials seem to be extremely involved.

Acknowledgments

The work of M.M. was supported, in part, by Fondecyt (Chile) Grant No. 1140568.

Appendix A: Generalised Gell-Mann matrices and basis transformations

Firstly, let us present a construction of the generalised Gell-Mann matrices λ_a of dimension n, that is, $a = 1, \ldots, n^2 - 1$. We start with defining the $n \times n$ matrix $\boldsymbol{e}_j \boldsymbol{e}_k^\dagger$ with a 1 in the jth row and kth column and 0 elsewhere. Here $\boldsymbol{e}_j$, $j = 1, \ldots, n$, are the Cartesian unit vectors in $\mathbb{C}_n$

$$\boldsymbol{e}_1 = \left(1,\ 0,\ \ldots,\ 0\right)^{\mathrm{T}}, \\ \vdots \\ \boldsymbol{e}_n = \left(0,\ \ldots,\ 0,\ 1\right)^{\mathrm{T}}, \tag{A1}$$

In terms of these matrices we construct $n^2 - 1$ hermitian traceless matrices $\lambda_a, a = 1, \ldots, n^2 - 1$ as follows. With $k = 1, \ldots, n-1$ and $j = 1, \ldots, k$ we set

$$\lambda_a = \boldsymbol{e}_j \boldsymbol{e}_{k+1}^\dagger + \boldsymbol{e}_{k+1} \boldsymbol{e}_j^\dagger, \quad \text{for } a = k^2 + 2j - 2, \tag{A2}$$

$$\lambda_a = -i\boldsymbol{e}_j \boldsymbol{e}_{k+1}^\dagger + i\boldsymbol{e}_{k+1} \boldsymbol{e}_j^\dagger, \quad \text{for } a = k^2 + 2j - 1. \tag{A3}$$

0	1 2	4 5	9 10	$\dots$	$(n-1)^2$ $(n-1)^2+1$
	3	6 7	11 12	$\dots$	$(n-1)^2+2$ $(n-1)^2+3$
		8	13 14	$\dots$	$(n-1)^2+4$ $(n-1)^2+5$
			15	$\dots$	$(n-1)^2+6$ $(n-1)^2+7$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$
				$\dots$	n^2-1

FIG. 1: Numbering scheme for the generalised Gell-Mann matrices λ_α ($\alpha = 0, \dots, n^2-1$).

In addition we construct $n-1$ diagonal matrices

$$\lambda_{(l+1)^2-1} = \sqrt{\frac{2}{l(l+1)}}\left[\left(\sum_{j=1}^{l} \boldsymbol{e}_j \boldsymbol{e}_j^\dagger\right) - l\cdot \boldsymbol{e}_{l+1}\boldsymbol{e}_{l+1}^\dagger\right], \tag{A4}$$

$$1 \le l \le n-1.$$

Eventually, we define the matrix λ_0, proportional to the unit matrix,

$$\lambda_0 = \sqrt{\frac{2}{n}}\mathbb{1}_n. \tag{A5}$$

Let us note that the matrices λ_α, ($\alpha = 0, \dots, n^2-1$) defined in this way in particular fulfill the conditions (2.7). An easy way to remember this numbering scheme is as follows. We draw an $n \times n$ square lattice and insert the numbers $\alpha = 0, 1, \dots, n^2-1$ as shown in Fig. 1. If α is the upper (lower) number in an off-diagonal square then λ_α gets a 1 ($-i$) in this place, 1 ($+i$) in the transposed place, and zero elsewhere. If α is in a diagonal square λ_α is given by (A4) for $\alpha > 0$ and by (A5) for $\alpha = 0$.

For $n = 3$ the matrices λ_a ($a = 1, \dots, 8$) as constructed above are the standard Gell-Mann matrices; see for instance [27].

Returning to the case of general n we find it convenient to define also

$$\begin{aligned}\lambda_+ &= \sqrt{\frac{n-1}{n}}\lambda_0 + \sqrt{\frac{1}{n}}\lambda_{n^2-1} = \sqrt{\frac{2}{n-1}}\begin{pmatrix}\mathbb{1}_{n-1} & 0\\ 0 & 0\end{pmatrix},\\ \lambda_- &= \sqrt{\frac{1}{n}}\lambda_0 - \sqrt{\frac{n-1}{n}}\lambda_{n^2-1} = \sqrt{2}\begin{pmatrix}0_{n-1} & 0\\ 0 & 1\end{pmatrix}.\end{aligned} \tag{A6}$$

The change form the basis $\lambda_0, \lambda_1, \dots, \lambda_{n^2-2}, \lambda_{n^2-1}$ to $\lambda_+, \lambda_1, \dots, \lambda_{n^2-2}, \lambda_-$ is made with help of the following orthogonal $n \times n$ matrix

$$\begin{aligned} S = (S_{\rho\alpha}) &= \begin{pmatrix} S_{+0} & 0 & S_{+,n^2-1}\\ 0 & \mathbb{1}_{n^2-2} & 0\\ S_{-0} & 0 & S_{-,n^2-1}\end{pmatrix}\\ &= \begin{pmatrix}\sqrt{\frac{n-1}{n}} & 0 & \sqrt{\frac{1}{n}}\\ 0 & \mathbb{1}_{n^2-2} & 0\\ \sqrt{\frac{1}{n}} & 0 & -\sqrt{\frac{n-1}{n}}\end{pmatrix}.\end{aligned} \tag{A7}$$

We have with $\alpha, \beta \in \{0, \dots, n^2-1\}$, $\rho, \sigma \in \{+, 1, \dots, n^2-2, -\}$

$$\begin{aligned}& S^{\mathrm{T}}S = SS^{\mathrm{T}} = \mathbb{1}_{n^2}, \qquad S = S^{\mathrm{T}},\\ & \lambda_\rho = S_{\rho\alpha}\lambda_\alpha,\\ & \mathrm{tr}(\lambda_\rho\lambda_\sigma) = 2\delta_{\rho\sigma}.\end{aligned} \tag{A8}$$

With the help of S we transform also K_α, ξ_α, $\tilde{E}_{\alpha\beta}$ (see (2.9), (3.2), (3.3)) to the basis $\rho, \sigma \in \{+, 1, \dots, n^2-2, -\}$

$$\begin{aligned}& K_\rho = S_{\rho\alpha}K_\alpha = \mathrm{tr}(\underline{K}\lambda_\rho),\\ & \xi_\rho = S_{\rho\alpha}\xi_\alpha,\\ & \tilde{E}_{\rho\sigma} = S_{\rho\alpha}\tilde{E}_{\alpha\beta}S^{\mathrm{T}}_{\beta\sigma}.\end{aligned} \tag{A9}$$

This gives, for instance, with (2.4)

$$\begin{aligned} K_+ &= \sqrt{\frac{n-1}{n}}K_0 + \sqrt{\frac{1}{n}}K_{n^2-1}\\ &= \sqrt{\frac{2}{n-1}}(\varphi_1^\dagger\varphi_1 + \ldots + \varphi_{n-1}^\dagger\varphi_{n-1}),\\ K_- &= \sqrt{\frac{1}{n}}K_0 - \sqrt{\frac{n-1}{n}}K_{n^2-1} = \sqrt{2}\varphi_n^\dagger\varphi_n.\end{aligned} \tag{A10}$$

Appendix B: Symmetric sums

Here we want prove the recursive relation (2.20) for the symmetric sums as originally defined in (2.12). Consider $1 \le k \le n$. First we note that $s_k(\kappa_1, \dots, \kappa_n)$ is a homogenous function of degree k in $\kappa_1, \dots, \kappa_n$. Therefore we have

$$\sum_{l=1}^{n} \kappa_l \frac{\partial}{\partial\kappa_l} s_k(\kappa_1, \dots, \kappa_n) = k s_k(\kappa_1, \dots, \kappa_n). \tag{B1}$$

On the other hand we have

$$\begin{aligned}\frac{\partial}{\partial\kappa_l}s_k(\kappa_1,\dots,\kappa_n) &= \sum_{\substack{1\le i_1<\ldots<i_{k-1}\le n\\ i_r\neq l}} \kappa_{i_1}\cdot\ldots\cdot\kappa_{i_{k-1}}\\ &= s_{k-1}(\kappa_1,\dots,\kappa_n) - \Big(\sum_{\substack{1\le i_1<\ldots<i_{k-2}\le n\\ i_r\neq l}}\kappa_{i_1}\cdot\ldots\cdot\kappa_{i_{k-2}}\Big)\kappa_l\\ &= s_{k-1}(\kappa_1,\dots,\kappa_n) - s_{k-2}(\kappa_1,\dots,\kappa_n)\kappa_l\\ &\quad + \ldots + (-1)^{k-1}s_0\kappa_l^{k-1}.\end{aligned} \tag{B2}$$

Multiplying in (B2) with κ_l, summing over l, and using (B1), we get

$$
\begin{aligned}
k s_k(\kappa_1, \ldots, \kappa_n) = {} & s_{k-1}(\kappa_1, \ldots, \kappa_n)(\kappa_1 + \ldots + \kappa_n) \\
& - s_{k-2}(\kappa_1, \ldots, \kappa_n)(\kappa_1^2 + \ldots + \kappa_n^2) \\
& + \ldots + (-1)^{k-1} s_0(\kappa_1^k + \ldots + \kappa_n^k) \quad \text{(B3)}
\end{aligned}
$$

which proves (2.20).

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